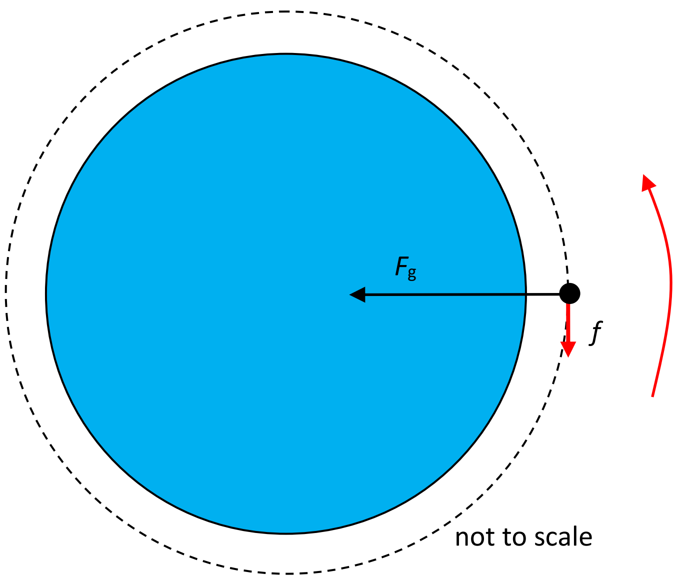
Teacher notes
Topic D

What is an estimate for the drag force on a satellite?

A satellite in low orbit experiences a drag force from the atmosphere. This force dissipates energy as thermal energy and so the **total energy** *E*, of the satellite is reduced. Since $E=-\frac{GMm}{2r}$, reducing *E* means making it more negative and so *r* becomes smaller; the satellite comes closer to the surface.

But the orbital speed in a circular orbit is given by $v=\sqrt{\frac{GM}{r}}$ and so reducing *r* means increasing the speed. This is the "satellite paradox": a frictional force ends up accelerating a body! Of course, there is no paradox: the kinetic energy increases but the gravitational potential energy decreases by more (twice in fact).



How can we estimate the magnitude of the drag force? How big do you guess it is?

At a height of 400 km a 5200 kg satellite has speed $v=\sqrt{\frac{GM}{r}}=\sqrt{\frac{6.67\times10^{-11}\times6\times10^{24}}{(6.4+0.4)\times10^{6}}}\approx7.7\times10^{3}\text{ m s}^{-1}$

and total energy $E=-\frac{GMm}{2r}=-\frac{6.67\times10^{-11}\times6\times10^{24}\times5200}{2\times(6.4+0.4)\times10^{6}}\approx-1.5\times10^{11}\text{ J}$. Typically, the satellite will reduce its height by about 1 km over a time of one month or a rate of $\frac{1.0\times10^3}{30\times24\times60\times60}\approx3.9\times10^{-4}\text{ m s}^{-1}$.

One full revolution takes $\frac{2\pi\times6.8\times10^6}{7.7\times10^3}\approx5.5\times10^3\text{ s}$ and so the orbit radius decreases by $\Delta r=5.5\times10^3\times3.9\times10^{-4}\approx2\text{ m}$ during one revolution. From uncertainties we know that:

$$\frac{\Delta E}{E}=-\frac{\Delta r}{r}\Rightarrow\Delta E=-E\frac{\Delta r}{r}=-(-1.5\times10^{11})\times\frac{-2}{6.8\times10^6}\approx-4.5\times10^4\text{ J}.$$

(If you do not like the uncertainty propagation argument you could calculate instead:
$\Delta E=-\frac{GMm}{2(r-\Delta r)}-(-\frac{GMm}{2r})$ with the same result.)

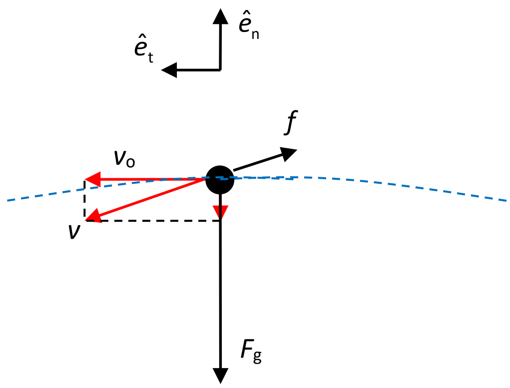
This loss of energy is due to the work done by the drag force and so we estimate, for one revolution:

$$f\times2\pi r=4.5\times10^4\text{ J, i.e. } f=\frac{4.5\times10^4}{2\pi\times6.8\times10^6}\approx0.001\text{ N}.$$

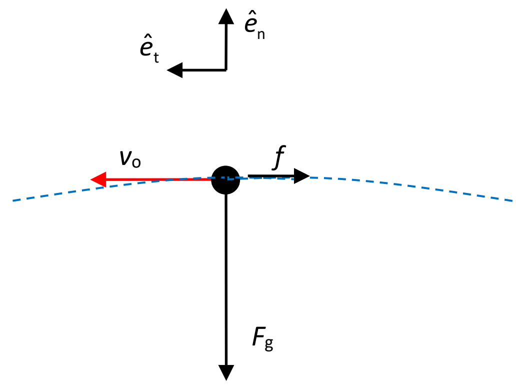
This is a tiny force, perhaps surprisingly so. It increases as the satellite mass increases. It also increases as the satellite falls because the density of air increases at lower heights. It increases further because the speed increases as the satellite falls. We should compare it to the gravitational force between the Earth and the satellite which is $\frac{GMm}{r^2}=\frac{6.67\times10^{-11}\times6\times10^{24}\times5200}{(6.8\times10^6)^2}\approx4.5\times10^4\text{ N}$.

Additional calculus-based material-optional.

The satellite is falling so the velocity acquires an additional component to the tangential orbital velocity component: $\vec{v}=v_o\hat{e}_t+\frac{dr}{dt}\hat{e}_n$ where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors tangential and radial to the path. (The radial component is the $3.9\times10^{-4}\text{ m s}^{-1}$ and the tangential component is the orbital speed of about $7.7\times10^3\text{ m s}^{-1}$ both as calculated above.)



Actual forces and velocities



Approximate forces and velocities

The rate of change of kinetic energy is equal to the power developed by the **net force** on the satellite and so

$$\begin{aligned}\frac{dE_K}{dt} &= \vec{F}\circ\vec{v} \\ &= (-f\hat{e}_t - F_g\hat{e}_n)\circ(v_o\hat{e}_t + \frac{dr}{dt}\hat{e}_n) \\ &= -fv_o - F_g\frac{dr}{dt} \\ &= -fv_o - \frac{GMm}{r^2}\frac{dr}{dt}\end{aligned}$$

From $E = -\frac{GMm}{2r}$ we find $\frac{dE}{dt} = \frac{GMm}{2r^2}\frac{dr}{dt}$. The rate of change of the **total** energy is equal to the power developed by the **external force** on the satellite i.e., the drag force. Hence

$$\frac{GMm}{2r^2}\frac{dr}{dt} = -fv \text{ or } \frac{GMm}{r^2}\frac{dr}{dt} = -2fv\,,$$

where v is the total speed of the satellite. Because the component $\frac{dr}{dt}\hat{e}_n$ is very small we may neglect it. In this case the drag force may be assumed to be acting in a direction opposite to the orbital velocity $v_o\hat{e}_t$ and the above equation becomes approximately:

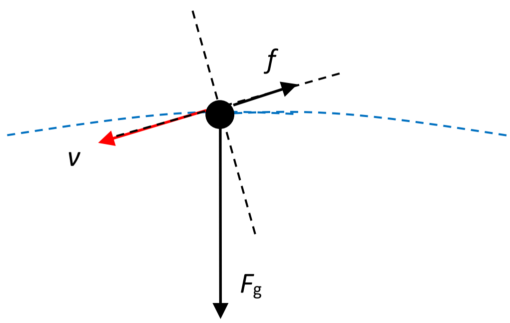
$$\frac{GMm}{r^2}\frac{dr}{dt} = -2fv_o$$

With this approximation:

$$\begin{aligned}\frac{dE_K}{dt} &= -fv_o + 2fv_o \\ &= +fv_o\end{aligned}$$

We see that even though the drag force acts opposite to the velocity, the kinetic energy and hence the speed of the satellite increases.

If you are still wondering how speed increases while a force opposite to velocity is acting, consider the following force diagram relative to axes along and normal to the velocity:



The much bigger gravitational force has a component *along* the velocity. This increases the satellite's speed while the frictional force decreases it. Overall, the speed increases.